Journal of Pure and Applied Algebra 30 (1983) 247-260
North-Holland

ABELIAN GROUPS IN A TOPOS: INJECTIVES AND INJECTIVE EFFACEMENTS

Roswitha HARTING
Mathematisches Institut, Universität Düsseldorf, Universitätsstraße 1, D-4000 Düsseldorf, Fed. Rep. of Germany

Communicated by C.J. Mulvey
Received 3 August 1982

Introduction

The starting point of this paper was the problem of the existence of enough injective abelian groups (in the following abbreviated by EIAG).

For ordinary abelian groups a natural proof of the existence of enough injectives proceeds by first showing that there are enough divisible abelian groups and then using the result of Baer that every divisible abelian group is injective. That this approach cannot work for the abelian groups in an arbitrary topos is already seen for the simplest case of a topos different from the usual topos of set theory, the topos $\mathrm{Shv}(X)$ of set-valued sheaves on a topological space X: In [6] Banaschewski shows for T_0-spaces X, that the injectivity of every divisible abelian group-valued sheaf on X (i.e. abelian group in $\mathrm{Shv}(X)$) implies X is discrete.

Further, in [4] Blass shows for Zermelo–Fraenkel set theory without the Axiom of Choice, that the injectivity of every divisible abelian group implies the Axiom of Choice. Moreover, in [3] Blass gives different models of Zermelo–Fraenkel set theory without the Axiom of Choice, among them one in which there are no non-trivial injective abelian groups, as well as another with EIAG. Furthermore we know that every Grothendieck topos (for example $\mathrm{Shv}(X)$) has EIAG [15]. Since a topos is an intuitionistic type theory ([7], [10], [12], [8], [19]), this means there are a lot of models of weak set theories that have EIAG. So the question arises what does it take for a model to have EIAG?

Examining the proofs which establish the existence of EIAG in certain abelian categories - such as Grothendieck categories with a generator e.g. the category of abelian groups in a Grothendieck topos - one notices that they depend on two facts: first that these categories have a *set* of generators and second that there are enough injective *ordinary* abelian groups.

Hence the appropriate context in which to approach the problem is to start with a base topos $\mathscr{S}$ (in the above cases this is just $\mathscr{S}$, the category of sets) and then to

0022-4049/83/$3.00

consider topoi $\mathscr{F}$ which are bounded over $\mathscr{E}$ ('bounded' internalizes the fact that there is a set of generators; for $\mathscr{E} = \mathscr{S}$, $\mathscr{F}$ bounded over $\mathscr{S}$ implies that $\mathscr{F}$ is a Grothendieck topos). Consequently the following natural conjecture arises:

Conjecture C. If a topos $\mathscr{F}$ is bounded over a base topos $\mathscr{E}$, then $\mathscr{F}$ has EIAG if $\mathscr{E}$ has.

One way to establish this conjecture would be to prove the following two assertions:

(i) If X is an object in $\mathscr{E}$, then $\mathscr{E}/X$, the category of X-indexed families of objects in $\mathscr{E}$, has EIAG if $\mathscr{E}$ has.

(ii) If j is a topology in $\mathscr{E}$, then $\mathrm{sh}_j(\mathscr{E})$, the full subcategory of $\mathscr{E}$ whose objects are the j-sheaves, has EIAG if $\mathscr{E}$ has.

Now recall the following frequently used lemma:

Lemma. If $G: \mathfrak{A}\to\mathfrak{B}$ *is a functor which has a faithful left adjoint respecting monomorphisms, then* $\mathfrak{B}$ *has enough injectives if* $\mathfrak{A}$ *has.*

Hence assertion (i) is true if the existence of a faithful left exact left adjoint to the pullback functor $X^*: \mathrm{Ab}(\mathscr{E})\to\mathrm{Ab}(\mathscr{E}/X)$ can be established ($\mathrm{Ab}(\mathscr{E})$ resp. $\mathrm{Ab}(\mathscr{E}/X)$ denotes the category of abelian groups in $\mathscr{E}$ resp. $\mathscr{E}/X$, and X^* assigns to an abelian group in $\mathscr{E}$ the corresponding constant family). Thus *the question about* EIAG *leads to the construction of the internal coproduct of abelian groups.* As an important step towards this, it is shown in [16] that the pullback functor $X^*: \mathrm{Ab}(\mathscr{E})\to\mathrm{Ab}(\mathscr{E}/X)$ has a left exact left adjoint $\bigoplus_X: \mathrm{Ab}(\mathscr{E}/X)\to\mathrm{Ab}(\mathscr{E})$ provided $\mathscr{E}$ has a natural number object. (Note that in intuitionistic type theory our usual conception of the coproduct of abelian groups as being a subobject of the direct product is false.)

The remaining crucial point – the faithfulness of the internal coproduct $\bigoplus_X: \mathrm{Ab}(\mathscr{E}/X)\to\mathrm{Ab}(\mathscr{E})$ – will be shown in Section 1. Assertion (ii) as it is formulated is an open question – but it is true, if 'injective' is replaced by "injective effacement", i.e. if it is changed to the following: If j is a topology in $\mathscr{E}$, then $\mathrm{Ab}(\mathrm{sh}_j(\mathscr{E}))$ has enough injective effacements if $\mathrm{Ab}(\mathscr{E})$ has. Hence in Section 2 a weaker version of Conjecture C will be proved: If a topos $\mathscr{F}$ is bounded over a topos $\mathscr{E}$ with a natural number object, then $\mathrm{Ab}(\mathscr{F})$ has enough injective effacements if $\mathrm{Ab}(\mathscr{E})$ has. Finally, Section 3 will deal with two transfer theorems: The existence of enough injectives from $\mathrm{Ab}(\mathscr{E})$ to $\mathrm{Ab}(\mathscr{E}^{\mathbf{C}})$ for an internal category $\mathbf{C}$ in $\mathscr{E}$, and the existence of enough injective hulls from $\mathrm{Ab}(\mathscr{E})$ to $\mathrm{Ab}(\mathrm{sh}_j(\mathscr{E}))$ for a topology j in $\mathscr{E}$.

I would like to thank Myles Tierney for drawing my attention to these questions and André Joyal for helpful discussions on the subject.

0. Background

For an elementary topos $\mathscr{E}$, the category $\mathrm{Ab}(\mathscr{E})$ of abelian groups in $\mathscr{E}$ is an abelian category [18, 8.11]. For an object X in $\mathscr{E}$, $\mathscr{E}/X$ is the category of X-indexed families of objects in $\mathscr{E}$; the pullback functor $X^*: \mathscr{E} \to \mathscr{E}/X$ which assigns to an object in $\mathscr{E}$ the corresponding constant family, has both a left and a right adjoint: $\sum_X : \mathscr{E}/X \to \mathscr{E}$ the 'internal disjoint sum' is left adjoint to X^*, and $\prod_X : \mathscr{E}/X \to \mathscr{E}$ the 'internal product' is right adjoint to X^*. It follows that X^* and its right adjoint $\prod_X$ are both left exact and thus respect abelian groups. Hence they can be lifted to a pair of adjoint functors between $\mathrm{Ab}(\mathscr{E})$ and $\mathrm{Ab}(\mathscr{E}/X)$. If $\mathscr{E}$ has a natural number object, then $X^*: \mathrm{Ab}(\mathscr{E}) \to \mathrm{Ab}(\mathscr{E}/X)$ also has a left exact left adjoint $\oplus_X : \mathrm{Ab}(\mathscr{E}/X) \to \mathrm{Ab}(\mathscr{E})$, the internal coproduct [16].

Note that for an abelian group B in $\mathscr{E}$, $\prod_X X^*B$ is canonically isomorphic to B^X, hence the abelian group structure of B^X is induced by that of B.

In the following $\mathscr{E}$ will always denote an elementary topos *with a natural number object* and X an object in $\mathscr{E}$. For topos theoretical terminology we follow [18].

1. The faithfulness of the internal coproduct

As indicated in the introduction, the crucial point in achieving the 'transfer theorem' for the existence of enough injective abelian groups (resp. injective effacements) is the faithfulness of the internal coproduct $\oplus_X : \mathrm{Ab}(\mathscr{E}/X) \to \mathrm{Ab}(\mathscr{E})$. So this chapter will be concerned with the proof of this fact.

In [16] we have shown that in intuitionistic type theory our usual conception of the coproduct of abelian groups as being a subobject of the direct product is misleading. Only for decidable indexing objects can we safely say that the coproduct has a natural homomorphism in the product. Hence there is no reason why the case $\mathscr{E} = \mathscr{S}$ should give a hint for the proof we are looking for. But again – as for the construction of the internal coproduct – we make use of the intriguing double nature, both set theoretical and geometrical, of topos theory. The geometrical mode will allow us to reduce the problem to the boolean case, and there the set theoretical mode enabled us to proceed 'as in Sets'.

1.1. Recall that an object X in $\mathscr{E}$ is called *decidable* if the diagonal $\Delta : X \rightarrowtail X \times X$ has a complement in $X \times X$.

1.2. Let $A(x)$ be an abelian group in $\mathscr{E}/X$. By using the set theoretical mode of topos theory it has been shown in [16, (1.10)] that for decidable X there is a monomorphism $m: A(x) \to X^* \prod_X A(x)$ in $\mathrm{Ab}(\mathscr{E}/X)$ that internalizes the canonical inclusion of the abelian group $A(x)$ into the product of abelian groups $\prod_{y \in X} A(y)$.

1.3. Proposition. *If X is decidable, then* $\oplus_X : \mathrm{Ab}(\mathscr{E}/X) \to \mathrm{Ab}(\mathscr{E})$ *is faithful.*

Proof. We have to show that the unit η of the adjunction $\oplus_X \dashv X^*$ is a monomorphism. So let $A(x)$ be an abelian group in $\mathscr{E}/X$. Since X is decidable it follows as mentioned above that there is a monomorphism $m: A(x) \rightarrowtail X^* \prod_X A(x)$. Because of the adjunction $\oplus_X - X^*$, m defines a morphism $m^{\#}: \oplus_X A(x) \to \prod_X A(x)$ such that $m = X^*(m^{\#})\eta_{A(x)}$. Hence $\eta_{A(x)}$ is a monomorphism since m is.

1.4. Corollary. *If $\mathscr{E}$ is a boolean topos, then $\oplus_X : \mathrm{Ab}(\mathscr{E}/X) \to \mathrm{Ab}(\mathscr{E})$ is faithful for every object X in $\mathscr{E}$.*

Proof. Follows immediately from (1.3) since in a boolean topos every object is decidable [1, 2.6].

Before showing how the general case reduces to the boolean one, we want to prove a very useful proposition. Note that the main part of it is a corollary of exercise 4.7 in [18], but we think the case given below valuable enough to be proved once explicitly.

1.5. Proposition. *Let $f: \mathscr{F} \to \mathscr{E}$ be a geometric morphism. Denote by η the unit of the adjunction $f^* - f_*$, by f_X the functor*

$$f_X : \mathscr{F}/f^*X \to \mathscr{E}/f_*f^*X; \qquad \left(F \xrightarrow{\alpha} f^*X\right) \mapsto \left(f_*F \xrightarrow{f_*\alpha} f_*f^*X\right)$$

and by ${}_Xf_*$ the following composite:

$$\begin{tikzcd} \mathscr{F}/f^*X \arrow[rr, dashed, "{}_Xf_*"] \arrow[dr, "f_X"'] & & \mathscr{E}/X \\ & \mathscr{E}/f_*f^*X \arrow[ur, "\eta_X^*"'] & \end{tikzcd}$$

(i) *The functor*

$${}_Xf^* : \mathscr{E}/X \to \mathscr{F}/f^*X; \qquad \left(E \xrightarrow{\alpha} X\right) \mapsto \left(f^*E \xrightarrow{f^*\alpha} f^*X\right)$$

is a left exact left adjoint to ${}_Xf_*$.

${}_Xf : \mathscr{F}/f^*X \to \mathscr{E}/X$ is thus a geometric morphism.

(ii) *The following two diagrams commute up to canonical isomorphism:*

$$\begin{tikzcd} \mathscr{F}/f^*X \arrow[r, "{}_Xf_*"] \arrow[d, dashed, "\coprod_{f^*X}"'] \arrow[dr, phantom, "(1)"] & \mathscr{E}/X \arrow[d, "\coprod_X"] \\ \mathscr{F} \arrow[r, "f_*"] & \mathscr{E} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}/f^*X \arrow[r, "{}_Xf_*"] \arrow[dr, phantom, "(2)"] & \mathscr{E}/X \\ \mathscr{F} \arrow[r, "f_*"] \arrow[u, "(f^*X)^*"] & \mathscr{E} \arrow[u, "X^*"'] \end{tikzcd}$$

(iii) *If f^* is a logical functor, then the following diagram commutes up to canonical isomorphism:*

$$\begin{tikzcd}
\mathcal{F}/f^*X \arrow[d, "\Pi_{f^*X}"'] \arrow[dr, phantom, "(3)"] & \mathcal{E}/X \arrow[l, "{}_Xf^*"'] \arrow[d, "\Pi_X"] \\
\mathcal{F} & \mathcal{E} \arrow[l, "f^*"']
\end{tikzcd}$$

Here obviously $\mathcal{E}$ does not need to have a natural number object.

Proof. (i) ${}_Xf^*$ is left adjoint to ${}_Xf_*$ since we have a sequence of natural bijections between the following sets of morphisms:

$$\begin{array}{ll}
{}_Xf^*\left(\begin{array}{c}E\\ \downarrow\alpha\\ X\end{array}\right) \to \begin{array}{c}F\\ \downarrow\beta\\ f^*X\end{array} \quad \text{in } \mathcal{F}/f^*X & \\
\hline
 & \text{by the definition of } {}_Xf^* \\
\begin{array}{c}f^*E\\ \downarrow f^*\alpha\\ f^*X\end{array} \to \begin{array}{c}F\\ \downarrow\beta\\ f^*X\end{array} & \\
\hline
 & \text{since } f^* \dashv f_* \\
\begin{array}{c}E\\ \downarrow\alpha\\ X\\ \downarrow\eta_X\\ f_*f^*X\end{array} \to \begin{array}{c}f_*F\\ \downarrow f_*\beta\\ f_*f^*X\end{array} \quad \text{in } \mathcal{E}/f_*f^*X & \\
\hline
\Sigma_{\eta_X}\left(\begin{array}{c}E\\ \downarrow\alpha\\ X\end{array}\right) \to \begin{array}{c}f_*F\\ \downarrow f_*\beta\\ f_*f^*X\end{array} & \\
\hline
 & \text{since } \Sigma_{\eta_X} \dashv \eta_X^* \\
\begin{array}{c}E\\ \downarrow\alpha\\ X\end{array} \to \eta_X^*\left(\begin{array}{c}f_*F\\ \downarrow f_*\beta\\ f_*f^*X\end{array}\right) \quad \text{in } \mathcal{E}/X & \\
\hline
 & \text{by the definition of } {}_Xf_* \\
\begin{array}{c}E\\ \downarrow\alpha\\ X\end{array} \to {}_Xf_*\left(\begin{array}{c}E\\ \downarrow\beta\\ f^*X\end{array}\right) &
\end{array}$$

Moreover ${}_Xf^*$ is left exact since f^* is.

(ii) All the functors in diagram (1) have a left adjoint. Hence diagram (1) commutes up to canonical isomorphism if the corresponding diagram of the left adjoints

$$\begin{tikzcd}
\mathcal{F}/f^*X & \mathcal{E}/X \arrow[l, "{}_Xf^*"'] \\
\mathcal{F} \arrow[u, "(f^*X)^*"] & \mathcal{E} \arrow[l, "f^*"'] \arrow[u, "X^*"']
\end{tikzcd}$$

commutes up to canonical isomorphism. This is the case since ${}_Xf^*$ is left exact, hence respects finite products.

Obviously the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}/f^*X \arrow[d, "\Sigma_{f^*X}"'] & \mathcal{E}/X \arrow[l, "{}_Xf^*"'] \arrow[d, "\Sigma_X"] \\
\mathcal{F} & \mathcal{E} \arrow[l, "f^*"']
\end{tikzcd}$$

Hence the corresponding diagram of the right adjoints – which is diagram (2) – commutes up to canonical isomorphism.

(iii) Recall that $\prod_X^E(\downarrow\alpha)$ may be obtained by the following pullback:

$$\begin{tikzcd}
\prod_X^E(\downarrow\alpha) \arrow[r] \arrow[d] & E^X \arrow[d, "\alpha^X"] \\
1 \arrow[r, "\ulcorner \mathrm{id}_X \urcorner"] & X^X
\end{tikzcd}$$

Since f^* is a logical functor it respects pullbacks and exponentiation, it thus follows that diagram (3) commutes up to canonical isomorphism.

1.6. Lemma. *Assume the same as in* 1.5. *If $f^*: \mathcal{E} \to \mathcal{F}$ is faithful, then*

$${}_Xf^*: \mathcal{E}/X \to \mathcal{F}/f^*X$$

is also faithful.

1.7. Theorem. $\oplus_X : \mathrm{Ab}(\mathcal{E}/X) \to \mathrm{Ab}(\mathcal{E})$ *is a faithful left exact left adjoint to* $X^*: \mathrm{Ab}(\mathcal{E}) \to \mathrm{Ab}(\mathcal{E}/X)$.

Proof. It remains to show that $\oplus_X : \mathrm{Ab}(\mathcal{E}/X) \to \mathrm{Ab}(\mathcal{E})$ is faithful. Now, from [18, 7.54] it follows that there exists a geometric morphism $f: \mathcal{F} \to \mathcal{E}$ where $\mathcal{F}$ is a Boolean topos such that $f^*: \mathcal{E} \to \mathcal{F}$ is faithful. Moreover it follows from 1.5(ii) that the following diagram commutes up to canonical isomorphism:

$$\begin{tikzcd}
\mathrm{Ab}(\mathcal{F}/f^*X) \arrow[d, "{}_Xf_*"'] & \mathrm{Ab}(\mathcal{F}) \arrow[l, "(f^*X)^*"'] \arrow[d, "f_*"] \\
\mathrm{Ab}(\mathcal{E}/X) & \mathrm{Ab}(\mathcal{E}) \arrow[l, "X^*"']
\end{tikzcd}$$

where the same notations as in 1.5 are used.

All the functors appearing in the diagram have a left adjoint. Hence the corresponding diagram of these left adjoints commutes too up to canonical isomorphism:

$$\begin{tikzcd}
\mathrm{Ab}(\mathcal{F}/f^*X) \arrow[r, "\oplus_{f^*X}"] & \mathrm{Ab}(\mathcal{F}) \\
\mathrm{Ab}(\mathcal{F}/X) \arrow[u, "{}_Xf^*"] \arrow[r, "\oplus_X"] & \mathrm{Ab}(\mathcal{E}) \arrow[u, "f^*"']
\end{tikzcd}$$

${}_Xf^*:\mathrm{Ab}(\mathcal{E}/X)\to\mathrm{Ab}(\mathcal{F}/f^*X)$ is faithful since f^* is (cf. 1.6). $\mathcal{F}$ is a Boolean topos, it thus follows from 1.4 that $\oplus_{f^*X}:\mathrm{Ab}(\mathcal{F}/f^*X)\to\mathrm{Ab}(\mathcal{F})$ is faithful. Hence $\oplus_{f^*X}\cdot {}_Xf^*$ is faithful.

But $f^*\oplus_X\simeq\oplus_{f^*X}\cdot {}_Xf^*$, therefore $f^*\oplus_X$ is faithful and consequently

$$\oplus_X:\mathrm{Ab}(\mathcal{E}/X)\to\mathrm{Ab}(\mathcal{E})$$

is faithful.

2. The existence of enough injective effacements in $\mathrm{Ab}(\mathcal{E})$

The notion of injective effacement was introduced by Grothendieck in [15] where it is shown that the construction of satellites in abelian cohomology does not really require the existence of enough injectives but only that of enough injective effacements.

Obviously for every Grothendieck topos $\mathcal{E}$, $\mathrm{Ab}(\mathcal{E})$ has enough injective effacements, since it actually has enough injectives, but this result is false for an arbitrary topos $\mathcal{E}$ (even with a natural number object). In [4] Blass gives an example of a topos $\mathcal{E}$ without any non-trivial injective abelian group, and it is easy to see that $\mathrm{Ab}(\mathcal{E})$ has no non-trivial injective effacements either. Concerning the results about the direct existence of enough injective effacements as for example in [2], [15], [20], note that they do not apply to the case of topoi, since they all assume some completeness and cocompleteness conditions not fulfilled by $\mathrm{Ab}(\mathcal{E})$. On the other hand, that these conditions are not necessary is shown by an example of a topos $\mathcal{E}$ given in [3]. Here $\mathrm{Ab}(\mathcal{E})$ has even enough injectives but only finite coproducts. (A description of this topos is also given in [5, Section 4].)

In the present section, some transfer principles for injective effacements are proved (compare also [20]) and they are used to show that the conjecture mentioned in the introduction is true for injective effacements, i.e. if a topos $\mathscr{F}$ is bounded over $\mathscr{E}$, then Ab($\mathscr{F}$) has enough injective effacements if Ab($\mathscr{E}$) has.

2.1. Recall that a monomorphism $k: B \rightarrowtail D$ in a category $\mathbb{A}$ is called an *injective effacement* if for every monomorphism $m: A \rightarrowtail C$ and every morphism $g: A \to B$ in $\mathbb{A}$ there is a morphism $f: C \to D$ in $\mathbb{A}$ such that $fm = kg$.

$\mathbb{A}$ is said to have *enough injective effacements* if for every object B in $\mathbb{A}$ there exist an object D and a monomorphism $k: B \rightarrowtail D$ in $\mathbb{A}$ such that $k: B \rightarrowtail D$ is an injective effacement in $\mathbb{A}$.

2.2. Lemma. *Let $G: \mathbb{A} \to \mathbb{B}$ be a functor which has a left adjoint $F: \mathbb{B} \to \mathbb{A}$ respecting monomorphisms.*

(i) *If F is faithful and if $\mathbb{A}$ has enough injective effacements, then so has $\mathbb{B}$.*

(ii) *If G is full and faithful and if $\mathbb{B}$ has enough injective effacements, then so has $\mathbb{A}$.*

Proof. (i) Suppose that F is faithful and that $\mathbb{A}$ has enough injective effacements. Let B be an object in $\mathbb{B}$. Then there is a monomorphism $k: FB \rightarrowtail A$ which is an injective effacement in $\mathbb{A}$. Since F is faithful it follows that $\eta_B: B \rightarrowtail GFB$, the unit of the adjunction $F \dashv G$ at B, is a monomorphism. Moreover G as a right adjoint respects monomorphisms, thus the composite $G(k)\eta_B: B \rightarrowtail GA$ is a monomorphism. It is now easy to check that this monomorphism is an injective effacement in $\mathbb{B}$.

(ii) Suppose that G is full and faithful and that $\mathbb{B}$ has enough injective effacements. Let A be an object in $\mathbb{A}$. Then there is a monomorphism $k: GA \rightarrowtail B$ which is an injective effacement in $\mathbb{B}$. Since F respects monomorphisms, $F(k): FGA \rightarrowtail FB$ is again a monomorphism. Now, G is full and faithful and therefore the counit ε of the adjunction $F \dashv G$ is an isomorphism. Hence the composite $F(k)\varepsilon_A^{-1}: A \rightarrowtail FB$ is a monomorphism. And again it is easy to check that this monomorphism is an injective effacement in $\mathbb{A}$.

2.3. Lemma. *Let $\mathbb{A}$ be an abelian category and $k: B \rightarrowtail D$ a monomorphism in $\mathbb{A}$. The following are equivalent:*

(i) *$k: B \rightarrowtail D$ is an injective effacement.*

(ii) *For every monomorphism $m: B \rightarrowtail C$ there is a morphism $f: C \to D$ such that $fm = k$.*

Proof. Suppose (ii) and let

$$\begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, dashed, "g"'] & C \\ B & \end{tikzcd}$$

be a diagram in $\mathbb{A}$ with m a monomorphism. Form the pushout of the given diagram to get the following diagram:

$$\begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, "g"'] & C \arrow[d, "l"] \\ B \arrow[r, tail, "n"] \arrow[dr, "k"'] & P \arrow[d, dashed, "h"] \\ & D \end{tikzcd}$$

Since $\mathbb{A}$ is an abelian category a push-out along a monomorphism is again a monomorphism, hence n is a monomorphism. We apply the assumption (ii) to get a morphism $h:P\to D$ such that $hn=k$. Thus for $f=hl$ we have $fm=kg$.

The other implication is trivial.

2.4. Remark. From [18, p. 120] we conclude that Giraud's theorem about the characterization of Grothendieck topoi can be reduced to the statement:

"A topos $\mathcal{F}$ is a Grothendieck topos iff it is defined over Sets and contains an object whose subobjects generate $\mathcal{F}$."

In "Change of base for toposes with generators" Diaconescu is led to replace here the particular topos Sets by an arbitrary base topos. This leads - as mentioned in the introduction - to the notion of 'bounded morphism'. The 'relative' version of Giraud's theorem is then the following ([11], also [18, 4.46]):

If $f:\mathcal{F}\to\mathcal{E}$ is a geometric morphism, then the following are equivalent:

(i) f is bounded.

(ii) There exist an internal category $\mathbf{C}$ in $\mathcal{E}$ and an inclusion $i:\mathcal{F}\to\mathcal{E}^{\mathbf{C}}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathcal{E} \\ & \mathcal{E}^{\mathbf{C}} \arrow[ur, "\lim_{\mathbf{C}}"'] & \end{tikzcd}$$

commutes up to canonical isomorphism. ($\mathcal{E}^{\mathbf{C}}$ denotes the topos of internal diagrams on $\mathbf{C}$, we may think of it as the category of functors from $\mathbf{C}$ to $\mathcal{E}$. i inclusion means that $i_*:\mathcal{F}\to\mathcal{E}^{\mathbf{C}}$ is full and faithful, or equivalently that there is a (unique) topology j in $\mathcal{E}^{\mathbf{C}}$ such that $\mathcal{F}$ is equivalent to $\mathrm{sh}_j(\mathcal{E}^{\mathbf{C}})$, the full subcategory of $\mathcal{E}^{\mathbf{C}}$ whose objects are the j-sheaves - by an equivalence identifying i_* with the embedding $\mathrm{sh}_j(\mathcal{E}^{\mathbf{C}})\to\mathcal{E}^{\mathbf{C}}$).

Observe that bounded morphisms which arise naturally in the geometric aspect of topos theory, appear in logic too. Bounded morphisms are in general not logical, hence do not respect the internal logic so, as one might expect, they play a role in independence proofs. For example in the topos-theoretical approach to Cohen's proof of the independence of the Continuum Hypothesis the bounded morphism $\mathrm{Sh}_{\neg\neg}(\mathcal{S}^{\mathbf{P}})\to\mathcal{S}$ is exploited, where P is a certain poset in $\mathcal{S}$ ([24], also [18, 9.56]).

In a further proof, given in boolean-valued set theory, the bounded morphism $\mathrm{Shv}(\mathbf{B}, C)\to\mathcal{S}$ is exploited, where B is a certain Boolean algebra and C the canonical Grothendieck topology on $\mathbf{B}$. Note, that $\mathrm{Shv}(\mathbf{B}, C)$ is equivalent to the category of Boolean-valued sets based on B ([17], [21], [22]). For other examples see [9], [14].

2.5. Theorem. *Let $f:\mathcal{F}\to\mathcal{E}$ be a bounded morphism. If* $\mathrm{Ab}(\mathcal{E})$ *has enough injective effacements, then so has* $\mathrm{Ab}(\mathcal{F})$.

Proof. Suppose that $\mathrm{Ab}(\mathcal{E})$ has enough injective effacements. Since f is bounded there exist – as stated in 2.4 – an internal category $\mathbf{C}$ in $\mathcal{E}$ and an inclusion $i:\mathcal{F}\to\mathcal{E}^{\mathbf{C}}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathcal{E} \\ & \mathcal{E}^{\mathbf{C}} \arrow[ur, "\lim_{\mathbf{C}}"'] & \end{tikzcd}$$

commutes up to canonical isomorphism.

From [18, 2.31] it follows that there is a left exact comonad G on $\mathcal{E}/C_0$ such that $\mathcal{E}^{\mathbf{C}}$ is isomorphic to the topos $(\mathcal{E}/C_0)_G$ of G-coalgebras (C_0 is the object-of-objects of $\mathbf{C}$). Moreover [18, 2.32 and 4.12(ii)] imply that there is a geometric morphism $f:\mathcal{E}/C_0\to(\mathcal{E}/C_0)_G$ such that f^* is faithful. Hence there is a geometric morphism $v:\mathcal{E}/C_0\to\mathcal{E}^{\mathbf{C}}$ such that v^* is faithful. The functor $v_*:\mathrm{Ab}(\mathcal{E}/C_0)\to\mathrm{Ab}(\mathcal{E}^{\mathbf{C}})$ has thus a left exact left adjoint v^* which is faithful.

Moreover, it follows from (1.7) that $C_0^*:\mathrm{Ab}(\mathcal{E})\to\mathrm{Ab}(\mathcal{E}/C_0)$ has a faithful left exact left adjoint. Now, $\mathrm{Ab}(\mathcal{E})$ has enough injective effacements. Hence, applying twice 2.2(i), we deduce that $\mathrm{Ab}(\mathcal{E}^{\mathbf{C}})$ has enough injective effacements. Furthermore, since $i:\mathcal{F}\to\mathcal{E}^{\mathbf{C}}$ is an inclusion, $i_*:\mathrm{Ab}(\mathcal{F})\to\mathrm{Ab}(\mathcal{E}^{\mathbf{C}})$ is full and faithful. It thus follows from 2.2(ii) that $\mathrm{Ab}(\mathcal{F})$ has also enough injective effacements.

3. The existence of enough injectives in $\mathrm{Ab}(\mathcal{E})$

Let $f:\mathcal{F}\to\mathcal{E}$ be a bounded geometric morphism and consider the decomposition of f described in 2.4:

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr, "i"'] & & \mathcal{E} \\ & \mathcal{E}^{\mathbf{C}} \arrow[ur, "\lim_{\mathbf{C}}"'] & \end{tikzcd}$$

As stated in the introduction it will be shown that $\mathrm{Ab}(\mathscr{E}^{\mathbf{C}})$ has enough injectives if $\mathrm{Ab}(\mathscr{E})$ has.

Now, recall that in Ab(Shv(X)), the category of abelian group-valued sheaves on a topological space X, there are enough injective hulls [13]; indeed, Banaschewski recently observed that this result carries over to topoi in the following sense: if j is a topology in $\mathscr{E}$, then $\mathrm{Ab}(\mathrm{sh}_j(\mathscr{E}))$ has enough injective hulls if $\mathrm{Ab}(\mathscr{E})$ has.

It is neither known whether there is a transfer theorem for enough injective hulls from $\mathrm{Ab}(\mathscr{E})$ to $\mathrm{Ab}(\mathscr{E}^{\mathbf{C}})$ nor whether there is one for enough injectives from $\mathrm{Ab}(\mathscr{E}^{\mathbf{C}})$ to $\mathrm{Ab}(\mathscr{E})$.

3.1. Recall the following lemma already mentioned in the introduction:

Lemma. *If a functor $G:\mathbb{A}\to\mathbb{B}$ has a left adjoint that respects monomorphisms and is faithful, then $\mathbb{B}$ has enough injectives if $\mathbb{A}$ has.*

3.2. Proposition. *If* $\mathrm{Ab}(\mathscr{E})$ *has enough injectives, then so has* $\mathrm{Ab}(\mathscr{E}/X)$.

Proof. $\oplus_X:\mathrm{Ab}(\mathscr{E}/X)\to\mathrm{Ab}(\mathscr{E})$ is a left exact left adjoint to $X^*:\mathrm{Ab}(\mathscr{E})\to\mathrm{Ab}(\mathscr{E}/X)$. Moreover we have shown in 1.7 that $\oplus_X$ is faithful. It thus follows from 3.1 that $\mathrm{Ab}(\mathscr{E}/X)$ has enough injectives if $\mathrm{Ab}(\mathscr{E})$ has.

3.3. Theorem. *If* $\mathrm{Ab}(\mathscr{E})$ *has enough injectives, then so has* $\mathrm{Ab}(\mathscr{E}^{\mathbf{C}})$ *for every internal category* $\mathbf{C}$ *in* $\mathscr{E}$.

Proof. Suppose that $\mathrm{Ab}(\mathscr{E})$ has enough injectives and let $\mathbf{C}$ be an internal category in $\mathscr{E}$. In the proof of 2.5 we have shown that there is a functor $\upsilon_*:\mathrm{Ab}(\mathscr{E}/C_0)\to\mathrm{Ab}(\mathscr{E}^{\mathbf{C}})$ which has a faithful left exact left adjoint. Now, $\mathrm{Ab}(\mathscr{E})$ has enough injectives, hence by 3.2 also $\mathrm{Ab}(\mathscr{E}/C_0)$ and therefore, using 3.1, it follows that $\mathrm{Ab}(\mathscr{E}^{\mathbf{C}})$ has enough injectives.

3.4. Recall that a monomorphism $k:A\rightarrowtail B$ in a category $\mathbb{A}$ is called an essential extension of A (or just essential), if every morphism $f:B\rightarrowtail C$, for which fk is a monomorphism, is itself a monomorphism.

An injective hull of A is an essential extension $k:A\rightarrowtail B$ of A with B injective.

3.5. Proposition (Banaschewski). *Let $F:\mathbb{A}\to\mathbb{B}$ be a functor which respects monomorphisms and has a full and faithful right adjoint G. Then $\mathbb{B}$ has enough injective hulls if $\mathbb{A}$ has.*

Proof. Suppose that $\mathbb{A}$ has enough injective hulls, let B be an object in $\mathbb{B}$ and $k:GB\rightarrowtail J$ an injective hull of GB in $\mathbb{A}$. Consider the following commutative diagram:

$$\begin{tikzcd} GB \arrow[r, "\eta_{GB}"] \arrow[d, tail, "k"'] & GFGB \arrow[d, tail, "GFk"] \\ J \arrow[r, "\eta_J"] & GFJ \end{tikzcd}$$

where η denotes the unit of the adjunction $F \dashv G$. Let ε denote the counit, then $(G\varepsilon_B)\eta_{GB} = \mathrm{id}_{GB}$, hence η_{GB} is a monomorphism. G and F respect monomorphisms, therefore GFk is also a monomorphism and it follows from the commutativity of the diagram above that $\eta_J k$ is a monomorphism, thus also $\eta_J : J \rightarrowtail GFJ$, since k is essential. Now J injective implies that η_J is a coretraction. Furthermore G is full, it thus follows from [23, 16.5.4] that η_J is an isomorphism and consequently GFJ is injective. Since G is full and faithfull and respects monomorphisms, it reflects injectives, hence FJ is injective in $\mathbb{B}$.

Moreover G full implies that there is a morphism $m : B \to FJ$ in $\mathbb{B}$ with $G(m) = \eta_J k$.

Since G is faithful and respects monomorphisms, it reflects essential extensions; η_J and k are essential thus also $G(m) = \eta_J k$. It follows that $m : B \rightarrowtail FJ$ is essential. FJ is injective and consequently $m : B \rightarrowtail FJ$ is an injective hull of B.

3.6. Corollary. *If j is a topology in $\mathscr{E}$, then* $\mathrm{Ab}(\mathrm{sh}_j(\mathscr{E}))$ *has enough injective hulls if* $\mathrm{Ab}(\mathscr{E})$ *has. (Here $\mathscr{E}$ is not supposed to have a natural number object.)*

Proof. Follows immediately from 3.5 since the canonical geometric morphism $i : \mathrm{sh}_j(\mathscr{E}) \to \mathscr{E}$ is an inclusion, hence $i_* : \mathrm{Ab}(\mathrm{sh}_j(\mathscr{E})$ is full and faithfull.

3.7. Remark. Examining the proof of 3.5 one notices that once η_J is established to be a monomorphism for injective J it turns out to be even an isomorphism. Hence for j a topology in $\mathscr{E}$, an injective j-separated object is already a j-sheaf. To get the transfer theorem for injectives from $\mathrm{Ab}(\mathscr{E})$ to $\mathrm{Ab}(\mathrm{sh}_j(\mathscr{E}))$ it would thus 'suffice' to show that every abelian group B in $\mathrm{sh}_j(\mathscr{E})$ can be embedded in a j-separated injective abelian group in $\mathscr{E}$.

3.8. Remark. Let us finally mention for completeness' sake that the internal coproduct $\bigoplus_X : \mathrm{Ab}(\mathscr{E}/X) \to \mathrm{Ab}(\mathscr{E})$ enables us also to show that injectivity is not only an invariant notion in $\mathrm{Ab}(\mathscr{E})$ for $\mathscr{E}$ a Grothendieck topos but for every topos $\mathscr{E}$ with a natural number object. Thus:

(i) An abelian group B in $\mathscr{E}$ is injective iff X^*B is injective in $\mathrm{Ab}(\mathscr{E}/X)$ for every object X in $\mathscr{E}$; and consequently

(ii) An abelian group B in $\mathscr{E}$ is injective iff B^X is injective in $\mathrm{Ab}(\mathscr{E})$ for every object X in $\mathscr{E}$.

Indeed, (i) follows immediately from the fact that $\bigoplus_X : \mathrm{Ab}(\mathscr{E}/X) \to \mathrm{Ab}(\mathscr{E})$ is a left exact left adjoint to $X^*: \mathrm{Ab}(\mathscr{E}) \to \mathrm{Ab}(\mathscr{E}/X)$, since this implies that X^* respects injectivity. Similarly it follows that $\prod_X : \mathrm{Ab}(\mathscr{E}/X) \to \mathrm{Ab}(\mathscr{E})$ respects injectivity. Therefore (ii) follows from (i), since B^X is canonically isomorphic to $\prod_X X^*B$.

Note that injectivity of abelian groups is not respected by logical funcors $f^*: \mathscr{E} \to \mathscr{F}$, even not if f^* is part of an essential geometric morphism $f: \mathscr{F} \to \mathscr{E}$, although f^* is then not 'too far away' from being a pullback functor. (If $f: \mathscr{F} \to \mathscr{E}$ is an essential geometric morphism with f^* logical, then the additional assumption that the left adjoint $f_!$ of f^* respects equalizers ensures that $\mathscr{F}$ is equivalent to $\mathscr{E}/X$ for some X by an equivalence identifying f^* with X^* [18, 1.47].)

Let us give an example: Let G be the abelian two-element group, $G = \{0, a\}$ where 0 is the neutral element. Then $\mathscr{S}^{G^{op}}$ is the topos of sets with a left action of the group G. Consider the essential geometric morphism $\gamma: \mathscr{S}^{G^{op}} \to \mathscr{S}$ defined by

$$\begin{aligned} &\gamma^*: \mathscr{S} \to \mathscr{S}^{G^{op}}; && S \mapsto S_G \text{ the set } S \text{ with trivial } G\text{-action}, \\ &\gamma_*: \mathscr{S}^{G^{op}} \to \mathscr{S}; && M \mapsto \text{the set of } G\text{-fixed elements of } M, \\ &\gamma_!: \mathscr{S}^{G^{op}} \to \mathscr{S}; && M \mapsto \text{the set of } G\text{-orbits of } M. \end{aligned}$$

$(\gamma_!, \gamma^*, \gamma_*)$ constitute an essential geometric morphism and γ^* is logical.

$\mathrm{Ab}(\mathscr{S}^{G^{op}})$ is the cateogry of left $\mathbb{Z}[G]$-modules; we want to show, that

$$\gamma^*: \mathrm{Ab} \to \mathbb{Z}[G]\text{-Mod}; \qquad A \mapsto A_G \text{ the abelian group } A \text{ with trivial } G\text{-action};$$

does not respect injectives.

Take M to be G with trivial G-action, N the cyclic group $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ with the G-action defined by $a\bar{1} = \bar{3}$. The morphism $m: M \to N$ defined by $m(a) = \bar{2}$ is a monomorphism in $\mathbb{Z}[G]$-Mod.

Let $k: M \to (\mathbb{Q}/\mathbb{Z})_G$ be defined by $k(a) = \overline{1/2}$, and suppose there is a morphism $f: N \to (\mathbb{Q}/\mathbb{Z})_G$ in $\mathbb{Z}[G]$-Mod such that

$$\begin{tikzcd} M \arrow[r, tail, "m"] \arrow[d, "k"'] & N \arrow[dl, "f"] \\ (Q/\mathbb{Z})_G & \end{tikzcd}$$

commutes. Now, $f(\bar{1}) = f(a\bar{3}) = af(\bar{3}) = f(\bar{3})$, hence

$$\overline{1/2} = k(a) = fm(a) = f(\bar{2}) = f(\bar{1}) + f(\bar{1}) = f(\bar{1}) + f(\bar{3}) = f(\bar{1} + \bar{3}) = \bar{0},$$

but $\overline{1/2} \neq \bar{0}$ in $\mathbb{Q}/\mathbb{Z}$.

Thus $\mathbb{Q}/\mathbb{Z}$ is injective in Ab, but $\gamma^*(\mathbb{Q}/\mathbb{Z}) = (\mathbb{Q}/\mathbb{Z})_G$ is not injective in $\mathbb{Z}[G]$-Mod.

References

[1] O. Acuña-Ortega and F.E.J. Linton, Finiteness and decidability I, in: Applications of Sheaves, Lecture Notes in Math. 753 (Springer, Berlin, 1979) 80–100.

[2] M. Barr, The existence of injective effacements, Canad. Math. Bull. 18 (1) (1975) 1–6.

[3] A. Blass, Fraenkel-Mostowski topoi, unpublished manuscript (1976).

[4] A. Blass, Injectivity, projectivity, and the axiom of choice, Trans. Amer. Math. Soc. 255 (1979) 31–59.

[5] B. Banaschewski, Extension of invariant linear functionals: Hahn-Banach in the topos of M-sets, J. Pure Appl. Algebra 17 (1980) 227–248.

[6] B. Banaschewski, When are divisible abelian groups injective? Quaestiones Math. 4 (1981) 285–307.

[7] A. Boileau, Types vs. topos, Thèse de Philosophiae Doctor, Université de Montréal (1975).

[8] A. Boileau et A. Joyal, La logique des topos, J. Symbolic Logic 46 (1981) 6–16.

[9] M. Bunge, Topos theory and Souslin's hypothesis, J. Pure Appl. Algebra 4 (1974) 159–187.

[10] M. Coste, Logique d'ordre supérieur dans les topos élémentaires, Séminaire Bénabou, Université Paris-Nord (1974).

[11] R. Diaconescu, Change of base for toposes with generators, J. Pure Appl. Algebra 6 (1975) 191–218.

[12] M.P. Fourman, The logic of topoi, in: J. Barwise, ed., Handbook of Mathematical Logic (North-Holland, Amsterdam, 1977) 1053–1090.

[13] P. Freyd, Abelian Categories (Harper and Row, New York, 1964).

[14] P. Freyd, The axiom of choice, J. Pure Appl. Algebra 19 (1980) 103–125.

[15] A. Grothendieck, Sur quelques points d'algèbre homologique, Tôhoku Math. J. 9 (1957) 119–221.

[16] R. Harting, Internal coproduct of abelian groups in an elementary topos, Commun. Algebra 10 (11) (1982) 1173–1237.

[17] D. Higgs, A category approach to boolean-valued set theory, Manuscript, Univ. of Waterloo, Ont. (1973).

[18] P.T. Johnstone, Topos Theory (Academic Press, New York, 1977).

[19] J. Lambek and P.J. Scott, Intuitionist type theory and the free topos, J. Pure Appl. Algebra 19 (1980) 215–257.

[20] P. Leroux, Sur les structures d'éffacement, Math. Z. 121 (1971) 329–340.

[21] G. Loullis, Sheaves and Boolean valued model theory, J. Symbolic Logic 44 (1979) 153–183.

[22] M.M. Richter, Geometrie und Logik 52 (1979), Schriften zur Informatik und Angewandten Mathematik, RWTH Aachen.

[23] H. Schubert, Categories (Springer, Berlin, 1972).

[24] M. Tierney, Sheaf theory and the Continuum Hypothesis, in: Lecture Notes in Math. 274 (Springer, Berlin, 1972) 13–42.